A Counterexample to the Rank-Coloring Conjecture

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ABSTRACT

It has been conjectured by C. van Nuffelen that the chromatic number of any graph with at least one edge does not exceed the rank of its adjacency matrix. We give a counterexample, with chromatic number 32 and with an adjacency matrix of rank 29.

All graphs in this note are finite and without loops or multiple edges. Let $G = (V, E)$ be a graph. A subset $X \subseteq V$ is *stable* if no edge of G has both ends in X. The *chromatic number* $\chi(G)$ of G is the minimum $k \geq 0$ such that V may be partitioned into k stable sets. The *adjacency matrix* of G is the matrix $M(G) = (m_{uv})_{u,v \in V}$ defined by $m_{uv} = 1$ if $u \neq v$ and u, v are adjacent, and $m_{uv} = 0$ otherwise. Let $rk(M(G))$ denote the rank of $M(G)$ over the real field.

It was conjectured by van Nuffelen in [4] in 1976, and later, independently, in [Fa] (by a computer program called Graffiti for generating conjectures in graph theory) that $\chi(G) \leq rk(M(G))$ for every graph G with at least one edge. This conjecture has recently attracted a considerable amount of interest, partly because it is related to an important open question in communication complexity [3]. Unfortunately, the conjecture is false; we shall give a counterexample with $\chi(G) = 32$ and $rk(M(G)) = 29$.

Let H be a *hypergraph*, that is, a set $V(H)$ together with a set $E(H)$ of subsets of $V(H)$. Assuming $\emptyset \in E(H)$, we construct a graph $G(H)$ from H as follows. Let W be the set of all functions from $V(H)$ into $GF(2)$. For each

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$A \in E(H)$, its *characteristic function* is the function mapping $v \in V(H)$ to 1 if and only if $v \in A$. Let L be the set of all characteristic functions of members of $E(H)$; then $0 \in L$. Put $K = W - L$. We define $G(H)$ to be the graph with vertex set W in which $u, v \in W$ are adjacent if $u \oplus v \in K$. (We use $\oplus$ to denote addition of vectors over $(GF(2)$.) Thus $G(H)$ is simply the Cayley graph of the group $(GF(2))^{|V(H)|}$ with respect to the set K, described here in detail for completeness. For two vectors $u, v \in W$, let $u \cdot v \in GF(2)$ denote their scalar product (over $GF(2)$).

Claim 1. The eigenvalues of $M(G(H))$ (with correct multiplicities) are the numbers $\sum_{k \in K} (-1)^{k \cdot w}$ $(w \in W)$.

Proof. This is a special case of a standard result about the eigenvalues of Cayley graphs of Abelian groups (see, for example, [2]) but we give a proof for the reader's convenience. For each $w \in W$, let

$$\lambda_w = \sum_{k \in K} (-1)^{k \cdot w}$$

and let x_w be the w-tuple whose uth term (for $u \in W$) is $(-1)^{u \cdot w}$. We claim that x_w is an eigenvector of $M(G(H))$ with eigenvalue λ_w. To see this, we observe that for each $v \in W$, the vth term of $M(G(H))x_w$ is

$$\sum [(-1)^{u \cdot w} : u \in W, u \oplus v \in K]$$

$$= \sum_{k \in K} (-1)^{(v \oplus k) \cdot w} = (-1)^{v \cdot w} \sum_{k \in K} (-1)^{k \cdot w} = \lambda_w (-1)^{v \cdot w}$$

and so $M(G(H))x_w = \lambda_w x_w$. Moreover, for distinct $w, w' \in W$,

$$x_w \cdot x_{w'} = \sum_{u \in W} (-1)^{u \cdot w} (-1)^{u \cdot w'} = \sum_{u \in W} (-1)^{u \cdot (w \oplus w)} = 0$$

since $w \oplus w' \neq 0$. Thus the x_w's form an orthogonal basis of eigenvectors, and the result follows. ∎

Claim 2. If $K \neq \emptyset$, the multiplicity of 0 as an eigenvalue of $M(G(H))$ is the number of subsets $X \subseteq V(H)$ such that $|X \cap A|$ is odd for precisely $\frac{1}{2}|E(H)|$ members $A \in E(H)$.

Proof. For each $w \in W$ define λ_w as before. If $w = 0$ then $\lambda_w \neq 0$ since $K \neq \emptyset$. If $w \neq 0$, then $k \cdot w$ is odd for precisely half of all $k \in W$; and $\lambda_w = 0$ if and only if $k \cdot w$ is odd for precisely half of all $k \in K$. We deduce that $\lambda_w = 0$ if and only if $k \cdot w$ is odd for precisely half of all $k \in L$. The claim follows from Claim 1. ∎

Now let H be the hypergraph with $V(H) = \{1, 2, \ldots, 6\}$ and

$$E(H) = \emptyset, \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1, 2, 3, 4, 5, 6\}\}.$$

The number of $X \subseteq V(H)$ such that $|X \cap A|$ is odd for precisely four members $A \in E(H)$ is 35 (namely, all 3-subsets and 4-subsets of $\{1, \ldots, 6\}$). By Claim 2 $M(G(H))$ has rank 29, since $G(H)$ has 64 vertices. Moreover, if $w_1, w_2, w_3 \in W$ form a stable set of size 3, then $w_1 \oplus w_2$, $w_2 \oplus w_3$, $w_3 \oplus w_1 \in L$, which is impossible since the sum of any three nonzero members of L is nonzero. We deduce that every stable set of $G(H)$ has cardinality ≤ 2, and so $\chi(G) \geq \frac{1}{2}|V(G)| = 32$. (Note that, in fact, $\chi(G) = 32$ as the complement of G contains a perfect matching).

It is easy and well known that for any graph G, $\chi(G) \leq 2^{rk(M(G))}$ (and in fact this inequality holds even when we let $rk(M(G))$ denote the rank of $M(G)$ over an arbitrary field). It remains open to decide if in general $\chi(G)$ is bounded by a polynomial in $rk(M(G))$—such a result would be of interest for [3]. Although one can easily produce from our example (by taking i disjoint copies of it and joining every vertex of each copy to every vertex of every other copy) a family of graphs $\{G_i\}_{i \geq 1}$ with $\chi(G_i) = 32i$ and $rk(M(G_i)) = 29i$, we cannot even show that $\chi(G)$ is not bounded by a linear function of $rk(M(G))$.

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